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Representation and Deduction

Abstract: Re-presentation, designating a re-play of conceptual structures, is contrasted with the popular idea of mental images of external things. The author claims that all deductive procedures require re-play of sensory-motor content as well as operational routines and demonstrates this in the classical syllogism and an example of simple addition. He concludes that the generation of deductive abilities in both logic and mathematics must be based on the practise of inductive inference.

...for the roses had the look
of flowers that are looked at
T. S. Eliot[1]

The epigraph was chosen from a poet, to make clear from the outset that my paper is intended as a contemplative stroll through theory and not as a report on empirical research. We are all concerned with teaching, and poets, I'm sure you will agree, have had some success in that area. They may have achieved it unintentionally, but their lack of didactic ambition does not seem to have impeded the learning of those who wanted to learn, nor has it diminished their desire to learn. I have no statistical evidence for these assertions, but I nevertheless intend to push further in the direction they indicate. Poets know, perhaps better than others, that readers or listeners cannot be given ready-made thoughts, images, and ideas. They can only be given words. Being given words, however, they will inevitably bring forth thoughts and images of their own; and by presenting particular combinations of words, one can, at least to a modest extent, guide the conceptual construction of the meaning which, eventually, the readers or listeners will believe to have found in the text.

Poets also know (only too well) that it is no use trying to tell a listener or reader that his or her interpretation is "wrong." Paul Valéry said:

> Once published, a text is like an appliance of which anyone can make use the way he likes and according to his means; it is not sure that the builder could use it better than others. Besides, he knows well what he wanted to make, and that knowledge always interferes with his perception of what he has made.[2]

Since teachers of mathematics, as a rule, know well what they are explaining, that knowledge invariably interferes with their own perception of their explanation. Consciously or unconsciously they take for granted that certain things are "self-evident," and they forget that what seems evident in mathematics is always contingent upon the habit of performing specific mathematical operations.

As seasoned users of language, we all tend to develop an unwarranted faith in the efficacy of linguistic communication. We act as though it could be taken for granted that the words we utter will automatically call forth in the listener the particular concepts and relations we intend to "express." We cling to the illusion that speech "conveys" ideas or mental representations. But words, be they spoken or written, do not convey anything. They can only call forth what is already there, and they can stimulate new combinations. This should become clear every time we test the representations our words have called forth in a listener; but we don't see it because of our unwarranted presuppositions concerning the process of "communication."

One misapprehension stems from the general notion of "representation." As that term is used in psychology and cognitive development, it is ambiguous in more than one way. First, like many words ending in "-ion," "representation" can indicate either an activity or its result. This ambiguity rarely creates difficulties. Far more serious is the epistemological ambiguity to which the word gives rise. It creates an unwholesome conceptual confusion.

The distinction I want to make clear concerns two concepts which, for instance in German, are expressed by two words, *Darstellung* and *Vorstellung*; both are usually rendered in English by "representation." The first designates an item that corresponds in an iconic sense to another item, an "original" to which it refers. The second designates a conceptual construct that has no explicit reference to something else of which it could be considered a replica or picture. (In fact, Vorstellung would be better translated into English as "idea" or "conception.")

Thus, if one uses the word in the second sense, it would help to spell it "re-presentation." The hyphenated "re" could be taken to indicate repetition of something one has experienced before. This would lessen the illusion that mental re-presentations are replicas or images of objects in some "real" world. It would help to focus attention on the fact that what one re-presents to oneself is never an independent external entity but rather the re-play of a conceptual item one has derived from experience by means of some sort of abstraction.[3]

The ability to re-present to oneself prior constructs is an essential part of all cognitive activities. It comes into play when you ask yourself whether the soufflé you are eating now is really as exquisite as the one you had in Dijon twenty years ago. Whenever you compare sensory experiences, and one of them is not in your actual perceptual field, that experience must be re-presented. As one gets older, one realizes that the memory from which one re-plays such past sensory experiences cannot always be trusted. Soufflés (and other sinful experiences) from one's distant past tend to seem sweeter than present ones. But this nostalgic tendency is not what I want to discuss here.

The ability to re-present is just as crucial in the use of symbols. The so-called "semantic nexus" that ties a symbol to what it is supposed to stand for, ceases to function when the symbol user is not able to re-present the symbol's meaning.

Memory, clearly, plays no less a part in the symbolic domain than in that of sensory experience.

Irrespective of the particular position you may have adopted concerning the foundations of mathematics, you will all agree that symbols such as "+," "–," "x," and ":" refer to operations and can, in fact, be interpreted as imperatives (add!, subtract!, multiply!, divide!). To obey any such imperative, one must not only "know" the operation it refers to, but also how to carry it out; one must know how to re-play the symbolized operation with whatever material happens to be at hand. That is to say, if operator-signs are to function as symbols, the operations to which they refer must have been abstracted by the symbol user from the sensory-motor material with which they were implemented in that symbol user's own prior experience. (There are, of course, several levels of abstraction, but at the bottom there has to be sensory-motor material.)

The semantic nexus between an operator-symbol and the abstracted operation it designates is no less indispensable in logic than it is in mathematics. Quine speaks of "the inseparability of the truths of logic from the meanings of the logical vocabulary."[4] Logical truth, of course, refers to the reliability of deductive inferences that can be derived from the chosen premises; it does not pertain to the experiential foundation of either premises or conclusions. If a syllogism were formed with the premises "All socialists are evil" and "Snoopy is a socialist" it would as logically lead to the conclusion that "Snoopy is evil" as the traditional syllogism leads to the conclusion that Socrates is mortal. The logical "truth" of a deduction is not impaired by the experiential falseness of the premises. Although the logic of the syllogism is in no way tied to what seems likely or unlikely in the thinker's experiential world, following the rules and carrying out the operations that are called forth by the use of words belonging to the "logical vocabulary" is nevertheless an activity and, as such, requires an active, thinking agent. Hume saw this, and concluded that deduction, because it involved a psychological process, could not be as infallible as classical logicians like to believe.[5] If this introduction of doubt were legitimate, doubt would eventually infest also the realm of mathematical operations. To discuss it may therefore not be an idle exercise—especially if, as I believe, Hume's notion can be tied to the theory of representation.

As far as deductive logic is concerned, what the premises say should always be explicitly posited rather than taken as statements of fact. Their relation to the experiential world is irrelevant. What matters is that they be taken as though they were unquestionable, as hypotheses which one accepts for the time being, and that their hypothetical status should always be carried over to the conclusion. In other words, we should always explicitly say:

IF all men are mortal,
and IF Socrates is a man,
Then Socrates is mortal.

This emphasizes two things: first, that one is dealing with assumptions whose experiential validity one has decided not to question for the moment; and, second, that the logical certainty one attributes to the conclusion pertains to the operations that are called forth by the logical terms "if," "all," "and," and "then." These two aspects are the basis of our faith in the infallibility of deductive procedures.

John Stuart Mill, in an attempt to subvert faith in the syllogism, argued that, in order truthfully to formulate the premise that all men are mortal, one should have to examine all members of the class called "men" with respect to their mortality. If, having done this, no exception to the rule had been found, one would know that Socrates is mortal, because, being a man, he must have been tested for mortality. If, on the other hand, he had not been tested, this could only mean that either he was not considered a "man," or that the use of the term "all" in the premise is unwarranted. This is a neat argument, but it shows that Mill did not see the premises of the syllogism as deliberate assumptions but as statements of experiential fact. Once this is understood, the argument no longer goes against the syllogism but against the misconception that deductive inferences should automatically be "true" in the experiential domain.

There may, however, be other problems. If the premises of syllogisms are understood as deliberately hypothetical conceptual structures (which one agrees not to question), one may still want to examine the deductive procedure, a procedure that involves several steps. Having constructed the premises, one must call up the logical operations designated by the tokens of the logical vocabulary and re-play these operations with the re-presentations of the premised conceptual structures. That is to say, in order to come to a conclusion, the conceptual construct created for the major premise must have been maintained unchanged, at least long enough to be available for re-presentation when one has created the conceptual construct for the minor premise and is ready to proceed with the logical operations that relate the two premises so as to produce the conclusion.

Whether or not one believes with Kant that the deductive operations called forth by logical terms are part of the inherent, a priori repertoire of the human mind, it seems plausible that, rather than being created each time anew, they are re-played, much like preprogrammed subroutines, when the associated symbol or sequence of symbols gains the agent's attention in an appropriate context.[6] If this is the case, some form of memory would be required for the performing of logical operations, and since memory would have to be considered a psychological phenomenon, one might be tempted to invoke Hume's doubt.

The question of memory arises even more clearly in connection with the hypothetical conceptual structures that are generated in response to the not specifically logical components of the premises, i.e., the hypothetical conceptual structures to which the logical operations must be applied. All deductive procedures require that we trust our ability to maintain, and re-present as they were, the conceptual structures and the operational routines we intend to use. If we doubt this ability, all logic goes by the board. We are not inclined that way. It would be as disruptive as doubting the reliability of memory and all the other electronic devices in a computer.

However, we may still question how we acquire logical operations. Professional philosophers usually dismiss any consideration of the developmental aspects of thought as "genetic fallacy" and pretend that logicians and other users of logical operations do not have to construct the required procedures but have them ready-made in their minds even if they do not always use them. Like Piaget, I find this an absurd contention. Instead, I would suggest that it is precisely the experiential success of inductively derived rules that provides both the occasions and the motivation for the abstraction of the specific logical operations that are then associated with symbols and used without reference to experience.

From that perspective, it seems clear that, in the construction of the syllogistic procedure, the components of the premises that are not the specifically logical terms must be interpretable by the active agent in a way that makes sense in the context of that agent's experience. It seems likely that we come to make the necessary reflective abstractions when we apply rules that work, rather than rules that are countermanded by experience. If we have never formulated a tentative rule of the kind "all roses I have seen, smelled sweet," we would not be tempted to say: "this flower looks like a rose—therefore it will smell sweet." In other words, if we have had no success with inductive inferences, we are unlikely to proceed to deductive ones.

To conclude, let me try to apply this line of thought to the basic understanding of numbers and how they interact. A child can no doubt learn by heart expressions such as "5 + 8 = 13." However, in order to understand them, she must be able to re-present the meanings of the involved symbols. As in the syllogism, the parts of such numerical expressions involve assumptions. "5" means that one assumes a plurality of countable items which, if they were counted (i.e. if number words were coordinated with them one-to-one), they would use up the number words from "one" to "five." The "+," then, signifies that a second plurality of items which, by itself, would use up the number words from "one" to "eight," is to be counted with the number words that follow upon "five."[7] Children may re-present these pluralities and the counting activity in many different ways. The sensory-motor material they use to implement the abstracted patterns is irrelevant. What matters is that they have abstracted these patterns and can re-play them in whatever context they might be needed. For I would claim that only if they have acquired a solid facility in the generation of this kind of representation can they possibly enter into the garden of mathematical delights.

Footnotes

1. Eliot, T. S. *Burnt Norton*, London: Faber & Faber, 1914; p.10.
2. Valery, P. Au sujet du cimetiere marin (1933). In: *Oeuvres de Paul Valery*, Paris: Galimard, La Pleiade, 1957; p.1507 (my translation).
3. Piaget, J. *La formation du symbole chez l'enfant* (1945). Neuchatel: Delachaux et Néstle, 7eme edition; p.237.
4. Quine, W. V. *Carnap and logical truth. In The ways of paradox.* Cambridge, Mass.: Harvard University Press, 1966; p.109.
5. Hume, D. *A treatise of human nature* (1739-40), Book I, Part IV, Ch. i.
6. Wittgenstein, L. *On certainty.* New York: Harper Torchbooks, 1969; 1522/523, p.69.
7. Steffe, L. P., von Glasersfeld, E., Richards, J. & Cobb, P. *Children's counting types: Philosophy, theory, and application.* New York: Praeger, 1983.

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