168 Glasersfeld E. von (1994) A radical constructivist view of basic mathematical concepts. In: Paul Ernest (ed.) Constructing mathematical knowledge: Epistemology and mathematics education. Falmer Press, London: 5–7. Available at http://www.vonglasersfeld.com/168

# A Radical Constructivist View of Basic Mathematical Concepts

I am not a mathematician. My interest is in conceptual analysis, and mathematical concepts are extremely interesting – especially the seemingly simple ones that are linked to the basic elements of arithmetic. The most basic of these elements are the symbols that we call ”numerals.”

There is an old statement that mathematics has to do with symbols and the manipulation of symbols. Frequent repetition of this statement has encouraged the belief that it is only the symbols that matter and that their conceptual referents need not be examined – presumably because to adults who have become used to doing arithmetic, there seems to be no difference between numerals and the concepts they refer to. But symbols do not generate the concepts that constitute their referents, they have to be linked to them by a thinking agent, even when this linkage has become automatic. It is, indeed, a ground rule of semiotics that a sound or a mark on paper becomes a symbol only when it is deliberately associated with a conceptual meaning.

I trust that you will agree that mathematics could not have happened if the concepts of “unit” and “Plurality of units” had not somehow been generated. How this was done may not be quite so obvious.

Thinkers as diverse as Edmund Husserl, Albert Einstein, and Jean Piaget have stated very clearly that the concept of “unit” is derived from the construction of “objects” in our experiential world.

Einstein’s description of this construction is one of the clearest:

> I believe that the first step in the setting of a “real external world” is the formation of the concept of bodily objects and of bodily objects of various kinds. Out of the multitude of our sense experiences we take, mentally and arbitrarily, certain repeatedly occurring complexes of sense impressions (partly in conjunction with sense impressions which are interpreted as signs for sense experiences of others), and we correlate to them a concept – the concept of the bodily object. Considered logically this concept is not identical with the totality of sense impressions referred to; but it is a free creation of the human (or animal) mind. (Einstein, 1954, p.291)

I have elsewhere shown that the concept of “plurality,” (unlike that of unitary physical object) cannot be derived from ’sense impressions’, but only from the awareness that the recognition of a particular physical object is being repeated. | 6

In Piaget’s terms, it is not an empirical abstraction from sensory-motor experience, but a reflective abstraction from the experiencer’s own mental operations. This is an important distinction. Let me try to make it as clear as possible:

Initially, the operations of constructing or recognizing physical objects do require sense impressions; but the realization that one is carrying out the same recognition procedure more than once, arises from one’s own operating and not from the particular sensory material that furnishes the occasions for this way of operating.

“Units” and “pluralities,” however, do not yet constitute a basis for the development of mathematics or even arithmetic. At least one other concept is needed: the concept of “number.”

In our book on *Counting Types*, Steffe, Richards, Cobb, and I have presented a model of how an abstract concept of “number” may be derived from the activity of counting. So far, I have not seen a more plausible model. But irrespective of the adequacy of that particular model, I have no doubt that both the notions of ordinal and cardinal number derive from the operations an active subject carries out and not from any specific sensory material.

If you agree that the concept of “number” requires both the concepts of “unit” and “plurality” – and if you further agree that *without* a concept of “number” there can be no development of a mathematics of numbers**,** then it is clear that this mathematics is an affair of *mental operations that have to be carried out by an active subject*. As Hersh says:

> Symbols are used as aids to thinking just as musical scores are used as aids to music. The music comes first, the score comes later. (Hersh, 1986, p.19)

If this is so, it would seem to follow incontrovertibly that a string of mathematical symbols remains meaningless until someone has associated specific mental operations with the symbols. These operations, because they are mental operations, cannot be witnessed by anyone else. What can be witnessed, are the symbols that acting subject produces in a spoken or written form as the result of his or her mental operations – and one can then examine whether or not they are compatible with the symbols one would have produced oneself.

This involves a two-fold process of *interpretation***:**

- First, the other subject’s interpretation of the string of symbols he or she has produced;
- Second, one’s own interpretation of these as one perceives them.

I submit that this awareness of processes of interpretation would have far-reaching consequences for any theory of proof. But this is not my present concern.

Here, I want to emphasize that the conceptual analyses I have suggested would have far-reaching consequences also for the teaching of arithmetic. If mathematical symbols have to be interpreted in terms of mental operations, the teacher’s task is to stimulate and prod the student’s mind to operate mathematically. Sensory-motor material, graphic representations, and talk can provide occasions for the abstraction of mathematical operations, but they cannot convey them ready-made to the student. The frequently used expression “mathematical objects” is misleading, because the meaning of mathematical symbols is never a static | object 7
but rather a particular way of operating. This goes for plurality and number, for point and line, and consequently for all the more complex abstractions of mathematics. The teacher’s task, then, is to orient the students mental operating; and to do this, the teacher needs at least a hypothetical model of how the student’s mind operates at the outset of the lesson (cf. Glasersfeld & Steffe, 1991).

To conclude, I would summarize these consequences by saying that, if we really want to teach arithmetic, we have to pay a great deal of attention to the mental operations of our students. Teaching has to be concerned with understanding rather than performance and the rote-learning of, say, the multiplication table, or training the mechanical performance of algorithms – because training is suitable only for animals whom one does not credit with a thinking mind.

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